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THE METRIC APPROXIMATION PROPERTY, NORM-ONE PROJECTIONS AND INTERSECTION PROPERTIES OF BALLS

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ABSTRACT

We show that if E is a Banach space with the Radon–Nikodym property, then E has the metric approximation property if and only if the space of finite rank operators is locally complemented in the space of bounded operators.

1. Introduction

In 1984 Harmand and Lima [9] proved that if E is a Banach space such that $K(E)$ is an M-ideal in $L(E)$, then E has the metric compact approximation property. A strong converse of this was proved by Cho and Johnson [3] in 1985 when they showed that if E is a subspace of ℓ_p, $1 < p < \infty$, with the compact approximation property, then $K(E)$ is an M-ideal in $L(E)$. Later on, D. Werner [25] extended Cho and Johnson's result to subspaces of c_0. Lately, Kalton [13] has obtained characterizations of separable spaces E such that $K(E)$ is an M-ideal in $L(E)$.

A projection P in a Banach space X is called an L-**projection** if

$$\|x\| = \|Px\| + \|x - Px\|$$

for all $x \in X$. A subspace M of a Banach space X is called an M-ideal if $M^\perp$ is the kernel of an L-projection in X^*. M-ideals were first studied by Alfsen and Effros [1]. They characterized M-ideals by means of intersection properties of balls. They proved that if M is a closed subspace of a Banach space X, then M is an M-ideal in X if and only if the following property holds:

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(#) whenever $\{B(a_i,r_i)\}_{i=1}^n$ is a finite family of balls in X such that $\bigcap_{i=1}^n B(a_i,r_i) \neq \emptyset$ and $M \cap B(a_i,r_i) \neq \emptyset$ for all i, then $M \cap \bigcap_{i=1}^n B(a_i,r_i+\epsilon) \neq \emptyset$ for all $\epsilon > 0$.

It suffices to consider 3 balls in the above intersection property.

The theorems by Harmand and Lima, Cho and Johnson, Werner and Kalton show that for some Banach spaces E we have equivalences between the following statements:

(i) E has the metric compact approximation property.

(ii) $K(E)^\perp$ is the kernel of an L-projection in $L(E)^*$.

(iii) $K(E)$ has the intersection property (#) in $L(E)$.

J. Johnson proved in [11] that if E is a Banach space with the metric approximation property, then $K(E)^\perp$ is the kernel of a norm-one projection in $L(E)^*$. This suggests that the results above might be generalized.

A subspace M of a Banach space X is said to have the $n.X.$ intersection property ($n.X.$I.P.) if whenever $\{B(a_i,r_i)\}_{i=1}^n$ is a family of balls in M such that $\bigcap_{i=1}^n B(a_i,r_i) \neq \emptyset$ in X, then

$$\bigcap_{i=1}^n B(a_i,r_i+\epsilon) \neq \emptyset \quad \text{in } M, \qquad \text{for all } \epsilon > 0.$$

The main result of this paper is the following result.

THEOREM 0: *Let E be a Banach space with the Radon–Nikodym property. Then the following statements are equivalent:*

(i) *E has the metric compact [metric] approximation property.*

(ii) *$K(E)^\perp$ [$R(E)^\perp$] is the kernel of a norm-one projection in $L(E)^*$.*

(iii) *$K(E)$ [$R(E)$] has the $n.L(E)$.I.P. for all n.*

Let us fix some notation. E, F, M, X and Y shall denote Banach spaces. The dual space of E is denoted E^*. If M is a subspace of E, then the annihilator of M in E^* is denoted $M^\perp$. $B(x,r)$ or $B_E(x,r)$ is the closed ball in E with center x and radius r.

We denote by $K(E,F)$ the space of compact linear operators from E to F, and by $L(E,F)$ the space of bounded linear operators from E to F. The space of finite rank operators from E to F is written $R(E,F)$. For $T \in L(E,F)$ we denote the adjoint operator by T^*.

The closure of a set S is written $\bar{S}$, and its convex hull is conv(S). Thus $\bar{R}(E,F)$ is the closure of $R(E,F)$ in $L(E,F)$. Unless otherwise stated the closure is taken in the norm topology. The set of extreme points of a convex set C is written ext C.

2. Complemented subspaces

We shall in this section assume that M is a closed subspace of a Banach space X. c_0 as a subspace of ℓ_∞ is an example of a non-complemented subspace such that its annihilator is the kernel of a norm-one projection in the dual space.

The following result is crucial in this paper. (See Fakhoury [6] and Kalton [12].) We shall indicate the main steps of its proof.

THEOREM 1: *If M is a closed subspace of a Banach space X, then the following statements are equivalent:*

(i) *$M^\perp$ is the kernel of a norm-one projection in X^*.*
(ii) *$M^{\perp\perp}$ is the image of a norm-one projection in X^{**}.*
(iii) *If F is a finite-dimensional subspace of X and $\epsilon > 0$, then there exists an operator $T : F \to M$ such that*
  (α) $x \in F \cap M \Rightarrow Tx = x$,
  (β) $\|T\| \le (1+\epsilon)$.

Proof: (i) $\Rightarrow$ (ii) follows by taking adjoints.
(ii) $\Rightarrow$ (iii). Let Q be the norm-one projection in X^{**} with $Q(X^{**}) = M^{\perp\perp}$. Let Q_F be the restriction of Q to F. Now we get T by composing Q_F with a suitable operator obtained by using "the principle of local reflexivity" [18].
(iii) $\Rightarrow$ (i). Here we use a compactness argument found in Lindenstrauss memoir [16].

For each finite-dimensional subspace F of X, choose an operator T_F as in (iii) with $\epsilon = 1/\dim F$. Let

$$S = \prod_{x \in X} B_{x^{**}}(0, 2\|x\|).$$

We equip S with the product weak*-topology. Then S is compact Hausdorff. For each F as above and each $x \in X$, we define

$$x_F = \begin{cases} T_F(x) & \text{if } x \in F, \\ 0 & \text{if } x \notin F. \end{cases}$$

$(x_F)_{x\in X}$ is a net in S ordered by $(x_F) > (x_G)$ if $G \subseteq F$. Let $(x_G)_{x\in X}$ be a subnet converging to a point $(y_x)_{x\in X}$ in S. For each $x \in X$ and each $x^* \in X^*$, we have that

$$x^*(x_G) \to y_x(x^*).$$

The map $x \to y_x$ from X to X^{**} is linear. For $x \in X$ and $x^* \in X^*$ define

$$(Px^*)(x) = y_x(x^*).$$

Then $P : x^* \to Px^*$ is a norm-one projection in X^* with kernel $M^\perp$. ∎

Remark: Theorem 1 is true if $\lambda \in [1, \infty\rangle$ and we replace "norm-one projection" and "$\|T\| \leq 1 + \epsilon$" by "projection of norm λ" and "$\|T\| \leq \lambda + \epsilon$". ∎

Definition: We shall say that M is **locally complemented** in X if (iii) in Theorem 1 is true. If (iii) is fulfilled with $\|T\| \leq \lambda + \epsilon$, then we shall say that M is **locally λ-complemented.** ∎

COROLLARY 2: *If X has the approximation property and M is locally λ-complemented in X for some $\lambda \in [1, \infty\rangle$, then M has the approximation property.*

Proof: Let $K \subseteq M$ be a compact set and let $\epsilon > 0$. There exists a finite rank operator $T : X \to X$ such that

$$\|Tx - x\| \leq \epsilon/(\lambda + \epsilon)$$

for all $x \in K$. Let F be the range of T and let $T_F : F \to M$ be as in (iii) of Theorem 1. Then $T_F T : M \to M$ is a finite rank operator and

$$\|T_F Tx - x\| \leq \epsilon$$

for all $x \in K$. ∎

Theorem 3 gives examples of locally complemented subspaces.

THEOREM 3: *Let E be a Banach space. Then $\bar{R}(E)$ is locally complemented in $\bar{R}(E^*)$. In particular, $\bar{R}(E)$ has the $n.\bar{R}(E^*)$.I.P. for all n.*

In this theorem we assume that $\bar{R}(E)$ is the closure in $\bar{R}(E^*)$ of the natural imbedding $T \to T^*$.

Proof: First we observe that in the proof of (iii) $\Rightarrow$ (i) in Theorem 1, we do not use that X is complete. Thus we only have to verify (iii) with $M = R(E)$ and $X = R(E^*)$. Let $F \subseteq R(E^*)$ be a finite-dimensional subspace and let $\epsilon > 0$. Let

$$G = \operatorname{span} \bigcup_{T \in F} T^*(E) \subseteq E^{**}.$$

Since each $T \in F$ is a finite rank operator, we get that G is finite dimensional.

By "the principle of local reflexivity", there exists a linear operator $V : G \to E$ such that

(α) $x \in G \cap E \Rightarrow Vx = x$,

(β) $|V| \leq (1 + \epsilon)$.

Define $U : F \to R(E)$ by

$$U(T) = V \circ T^*|_E.$$

Then $\|U\| \leq 1 + \epsilon$.

It easily follows that $U(T) = T$ for all $T \in F \cap R(E)$. ∎

Problem 1: *Is $K(E)$ locally complemented in $K(E^*)$?*

It is clear that M-ideals are locally complemented subspaces. We shall now modify (iii) in Theorem 1 to obtain a characterization of M-ideals.

Theorem 4: *Let M be a closed subspace of a Banach space X. The following statements are equivalent:*

(i) *M is an M-ideal in X.*

(ii) *If $F \subseteq X$ is a finite-dimensional subspace and $\epsilon > 0$, then there exists a linear operator $T : F \to M$ such that*

(α) *$x \in F \cap M \Rightarrow Tx = x$,*

(β) *$\|Tx + (I - T)y\| \leq (1 + \epsilon)\max(\|x\|, \|y\|)$,*

for all $x,\ y \in F$.

Proof: (ii) $\Rightarrow$ (i). Let $y \in B_X(0,1)$ and let $x_1,\ x_2,\ x_3 \in B_M(0,1)$. Let $\epsilon > 0$ and let $F = \text{span}\,(y, x_1, x_2, x_3)$. Let T be as in (ii). Then we have

$$Ty \in M \cap \bigcap_{i=1}^{3} B(y + x_i,\ 1 + \epsilon).$$

Thus M is an M-ideal in X [14].

(i) $\Rightarrow$ (ii). Let $\epsilon > 0$ and let $F \subseteq X$ be a finite-dimensional subspace. Since M is an M-ideal in X, we get that M is an M-ideal in $M + F$. Thus we can and shall assume that

$$\dim(X/M) < \infty.$$

Let Q be the M-projection in X^{**} onto $M^{\perp\perp}$. Let

$$H = Q(F) + (I - Q)(F) \subseteq X^{**}.$$

We have $\dim H < \infty$. By "the principle of local reflexivity", there exists an operator $S : H \to X$ such that

(a) $Sx = x$ for all $x \in H \cap X$,

(b) $x^*(x) = x^*(Sx)$ for all $x \in H$, all $x^* \in M^\perp$,

(c) $(1-\epsilon)\|x\| \leq \|Sx\| \leq (1+\epsilon)\|x\|$ for all $x \in H$.

If $x \in F \cap M \subseteq H \cap M^{\perp\perp} \cap X$, then

$$x = Qx = SQx$$

and (α) follows. From (b) it follows that $SQx \in M$ for all $x \in F$.

Let x, $y \in F$. Then

$$\begin{aligned} y &= Qy + (y - Qy) \\ &= SQy + S(y - Qy) \end{aligned}$$

so

$$y - SQy = S(y - Qy).$$

Hence

$$\begin{aligned} (1+\epsilon)\max(\|x\|, \|y\|) &\geq (1+\epsilon)\max(\|Qx\|, \|y - Qy\|) \\ &\geq \|SQx + S(y - Qy)\| \\ &= \|SQx + (y - SQy)\|. \end{aligned}$$

Put $T = SQ$, and the proof is complete. ∎

We shall now give an application of the previous theorem. Again we consider $R(E)$ to be naturally embedded into $R(E^*)$. Results similar to Theorem 5 for spaces of compact and bounded operators can be found in Corollary 2.4 and the Remark following Corollary 2.4 in [24].

THEOREM 5: *Let E and F be Banach spaces such that $E \subseteq F$. The following statements are equivalent:*

(i) *E is an M-ideal in F.*

(ii) *$\bar{R}(E, E)$ is an M-ideal in $\bar{R}(E, F)$.*

(iii) *$\bar{R}(X, E)$ is an M-ideal in $\bar{R}(X, F)$ for every Banach space X.*

In particular, E is an M-ideal in E^{**} if and only if $\bar{R}(E)$ is an M-ideal in $\bar{R}(E^*)$.

Proof: The last part of the theorem follows from the first part by identifying $R(E, E^{**})$ with $R(E^*)$.

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Let $x \in E$ and $y \in F$ with $1 = \|x\| = \|y\|$. Choose $x^* \in E^*$ such that $1 = x^*(x) = \|x^*\|$. Define $S \in R(E,E)$ and $T \in R(E,F)$ by $Sz = x^*(z)x$ and $Tz = x^*(z)y$. Both S and T have norm 1. Let $\epsilon > 0$. By (ii) there exists $U \in R(E,E)$ such that

$$\|S \pm (T - U)\| \leq 1 + \epsilon.$$

Then evaluating on x gives

$$\|x \pm (y - Ux)\| \leq 1 + \epsilon.$$

This shows that E is a semi M-ideal in F. The same argument using three balls gives that E is an M-ideal in F.

(i) $\Rightarrow$ (iii). Let X be a Banach space and let S_1, S_2 and S_3 be finite rank operators from X into E with norm ≤ 1. Let $T \in R(X,F)$ be a finite rank operator of norm one and let $\epsilon > 0$. Let H be a finite-dimensional subspace of F which contains the images of all these four operators. By Theorem 4, we can find an operator $U : H \to E$ such that $U(x) = x$ for all $x \in E \cap H$ and

$$\|Ux + (y - Uy)\| \leq (1+\epsilon)\max(\|x\|, \|y\|),$$

for all $x, y \in H$. Let $V = UT$. Then $V \in R(X,E)$ and

$$\|S_i + T - V\| \leq 1 + \epsilon$$

for $i = 1, 2, 3$. ∎

Remark: D. Werner has informed me that we cannot add

$$\bar{R}(E) \text{ is an } M\text{-ideal in } \bar{R}(E^{**})$$

to the last part of Theorem 5. In fact,

(*) if $\bar{R}(E)$ is a semi M-ideal in $\bar{R}(E^{**})$, then E is reflexive.

Proof of (*): Let $x^* \in E^*$ and $x^{***} \in E^{***}$ with $1 = \|x^*\| = \|x^{***}\|$. Choose $x \in E$, $y^* \in E^*$ and $x^{**} \in E^{**}$ such that

$$1 = \|x\| = \|y^*\| = \|x^{**}\| = y^*(x) = x^{**}(y^*).$$

Define $S \in R(E)$ and $T \in R(E^{**})$ by

$$S(y) = x^*(y)x,$$
$$T(y^{**}) = x^{***}(y^{**})x^{**}.$$

Then $1 = \|S\| = \|T\|$.

If $\epsilon > 0$ and $U \in R(E)$ such that

$$\|S^{**} \pm (T - U^{**})\| \leq 1 + \epsilon,$$

then taking adjoints and applying on y^* gives

$$\|x^* \pm (x^{***} - U^* y^*)\| \leq 1 + \epsilon.$$

Thus E^* is a semi M-ideal in E^{***}, hence an M-ideal. But this implies that E is reflexive [10]. ∎

It is easy to give examples of spaces which are locally complemented but not complemented. If M is an L_1-predual space which is a subspace of a space X, then $X^{\perp\perp} = \operatorname{im} P$ for a norm-one projection P in X^{**}. Thus M is locally complemented in X.

If X/M is an L_1-space, then $M^{\perp} = \operatorname{im} P$ for some norm-one projection P in X^*.

We have a theorem similar to Theorem 1 for quotient spaces.

THEOREM 6: *Let M be a closed subspace of a Banach space X. The following statements are equivalent.*

(i) *$M^{\perp}$ is the range of a norm-one projection in X^*.*

(ii) *$M^{\perp\perp}$ is the kernel of a norm-one projection in X^{**}.*

(iii) *If F is a finite-dimensional subspace of X and $\epsilon > 0$, then there exists an operator $T : F \to M$ such that*

$$Tx = x \quad \text{for all } x \in F \cap M, \quad \text{and}$$
$$\|I - T\| \leq 1 + \epsilon \quad \text{for all } x \in F.$$

(iv) *If G is a finite-dimensional subspace of X/M and $\epsilon > 0$, then there exists an operator $S : G \to X$ such that $\|S\| \leq 1 + \epsilon$ and $q \circ S$ is equal to the identity on G where $q : X \to X/M$ is the quotient map.*

Short proof:
(i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii) is similar to the proof of (i) $\Rightarrow$ (ii) in Theorem 4.
(iii) $\Rightarrow$ (i) is similar to (iii) $\Rightarrow$ (i) in Theorem 1.
(iii) $\Rightarrow$ (iv) is easy. ∎

Remark: (iv) in Theorem 6 is a local lifting. Just as there are many locally complemented subspaces which are not complemented, there are many quotient spaces which allow local liftings but not global liftings. Peter Harmand has made me aware of the following example.

Since $L_1 = L_1(0,1)$ is separable, there exists a closed subspace M of ℓ_1 such that

$$L_1 \simeq \ell_1/M \quad \text{(isometric)}.$$

Since no subspace of ℓ_1 is isometric to L_1, there is no lifting from L_1 to ℓ_1.

We have

$$L_1^* \simeq M^\perp \simeq C(K)$$

for some compact Hausdorff space K, so $M^\perp = \text{im } P$ for a norm-one projection P in $\ell_1^* = \ell_\infty$.

Note that L_1 has the metric approximation property. ∎

It is a trivial observation that if M is locally complemented in X, then M has the $n.X$.I.P. for all n. Let

$$M^\# = \{x^* \in X^* : \|x^*\| = \|x^*\|_M\}.$$

In Lima [15] it is proved that $M^\#$ is a linear subspace of X^* if and only if M has the $n.X$. I.P. for all n and, moreover, every $x^* \in M^*$ has a unique normpreserving extension to X. The $n.X$.I.P. for all n can be replaced by the $3.X$.I.P. (For a correction to Lima [15], see Oja [19].)

Let us examine the following example. If $x^* \in M^*$, denote by $HB(x^*)$ the w^*-compact convex set of normpreserving extensions of x^* to X.

Example: Let $X = \ell_\infty^4$. Define a map $T : \ell_1^3 \to X$ by

$$T(1,0,0) = (1,-1,-1,1),$$
$$T(0,1,0) = (-1,1,-1,1),$$
$$T(0,0,1) = (-1,-1,1,1),$$

and extend T to a linear operator.

Let $M = T(\ell_1^3)$.

We have:

(α) M has the 3.X.I.P., but not the 4.X.I.P.

(β) Every $x^* \in \text{ext } B_{M^*}(0,1)$ has a unique norm-preserving extension to X.

(γ) There exists $x^* \in M^*$ which does not have a unique norm-preserving extension to X.

M has the 3.X.I.P. since L_1-spaces have the 3.2.I.P. (See Lima [14].) Since L_1-spaces do not have the 4.2.I.P., we get that M does not have the 4.X.I.P.

Note that $M^\perp = \text{span}\,(1,1,1,1)$. Let $x^* \in \text{ext } B_{M^*}(0,1)$. Then $HB(x^*)$ is a face of $B_{X^*}(0,1)$ which is a point or a line-segment parallel to $M^\perp$. Since no face of $B_{X^*}(0,1)$ is parallel to $(1,1,1,1)$, we get that $HB(x^*)$ must be a point.

$(-1,-1,0,0,)$ and $(0,0,1,1)$ are two different normpreserving extensions of the functional $x^* \in M^*$ defined by

$$x^*(1,-1,-1,1) = 0,$$
$$x^*(-1,1,-1,1) = 0,$$
$$x^*(-1,-1,1,1) = 2.$$

3. Extensions of extreme functionals

As shown by Ruess and Stegall [20], we have

$$\begin{aligned}\text{ext } B_{R(E,F)^*}(0,1) &= \text{ ext } B_{K(E,F)^*}(0,1)\\ &= \text{ ext } B_{E^{**}}(0,1) \otimes \text{ ext } B_{F^*}(0,1).\end{aligned}$$

Note that for $T \in L(E,F)$ and $x^{**} \otimes x^* \in E^{**} \otimes F^*$, we define $(x^{**} \otimes x^*)(T) = x^{**}(T^*x^*)$.

For the dual of $L(E,F)$, we have the following result.

THEOREM 7: *For any Banach spaces E and F, we have*

$$B_{L(E,F)^*}(0,1) = \overline{\text{conv}}(\text{ext } B_{E^{**}}(0,1) \otimes \text{ext } B_{F^*}(0,1))$$

(weak-closure).*

This is proved in [10].

Often, we have $E = F$ and then the state space S_E is a useful object to study:

$$S_E = \{\phi \in L(E)^* : \|\phi\| = 1 = \phi(I)\}.$$

I is the identity operator on E.

Let

$$S_e = \{x^{**} \otimes x^* \in \text{ ext } B_{E^{**}}(0,1) \otimes \text{ ext } B_{E^*}(0,1) : x^{**}(x^*) = 1\}.$$

Then

$$S_e \subseteq S_E.$$

THEOREM 8: *For every Banach space E, we have*

$$S_E = \overline{\text{conv}}(S_e) \quad (\textit{weak}^*\textit{-closure in } L(E)^*).$$

Proof: Let $\phi \in S_E$ and let $\epsilon > 0$.

Choose $T_0, T_1, \ldots, T_n \in L(E)$ with $T_0 = I$ and all $\|T_j\| \leq 1$. Note that

$$B_{L(E)^*}(0,1) = \overline{\text{conv}}\{B_E(0,1) \otimes B_{E^*}(0,1)\}$$

(weak*-closure), so we can find $m, \lambda_i > 0$ and $x_i \otimes x_i^* \in B_E(0,1) \otimes B_{E^*}(0,1)$ such that $\Sigma_{i=1}^m \lambda_i = 1$ and

$$|\phi(T_j) - \sum_{i=1}^m \lambda_i x_i^*(T_j x_i)| < \partial^2$$

for $\partial = \epsilon^2/4$ and $j = 0, 1, \ldots, n$.

Define

$$J_0 = \{i : x_i^*(x_i) \leq 1 - \partial\}.$$

Since $\phi \in S_E$ and $T_0 = I$, we have

$$\begin{aligned} 1 - \partial^2 &< \sum_{i=1}^m \lambda_i x_i^*(x_i) \\ &\leq \sum_{i \in J_0} \lambda_i (1 - \partial) + \sum_{i \notin J_0} \lambda_i \\ &= 1 - \partial \sum_{i \in J_0} \lambda_i. \end{aligned}$$

Hence $\Sigma_{i \in J_0} \lambda_i < \partial$.

By deleting those elements with $x_i^*(x_i) \leq 1 - \partial$ and replacing λ_i by $\lambda_i \cdot (1 - \Sigma_{i \in J_0} \lambda_i)^{-1}$, we may assume $x_i^*(x_i) > 1 - \partial$ for all i and

$$|\phi(T_j) - \sum_{i=1}^m \lambda_i x_i^*(T_j x_i)| < \partial^2 + 3\partial$$

for all j. Clearly, we may assume $\|x_i^*\| = \|x_i\| = 1$ for all i.

Using the Bishop–Phelps–Bollobas theorem [2], and since $|x_i^*(x_i) - 1| < \partial$, there exist $y_i^* \in B_{E^*}(0,1)$ and $z_i \in B_E(0,1)$ such that

$$y_i^*(z_i) = 1, \quad \|y_i^* - x_i^*\| < \epsilon \quad \text{and} \quad \|x_i - z_i\| < \epsilon$$

for all i. Thus

$$\begin{aligned}|x_j^*(T_j x_i) - y_i^*(T_j z_i)| &\le |(x_i^* - y_i^*)(T_j^* x_i)| + |T_j^* y_i^*(x_i - z_i)| \\ &\le 2\epsilon\end{aligned}$$

for all i and all j. Hence

$$\Big|\phi(T_j) - \sum_{i=1}^{m} \lambda_i y_i^*(T_j z_i)\Big| < \partial^2 + 3\partial + 2\epsilon < 3\epsilon$$

for all j.

For each i, put

$$HB(z_i) = \{y^* \in B_{E^*}(0,1) : y^*(z_i) = 1\}.$$

$HB(z_i)$ is a w^*-closed face containing y_i^*. Since

$$HB(z_i) = \overline{\text{conv}}(\text{ext} HB(z_i)) \quad (\text{ weak}^*\text{-closure}),$$

we can replace each y_i^* by a convex sum of elements from ext $HB(z_i)$. By renumbering, we may assume each $y_i^* \in$ ext $B_{E^*}(0,1)$ and

$$\Big|\phi(T_j) - \sum_{i=1}^{m} \lambda_i y_i^*(T_j z_i)\Big| < 3\epsilon$$

for all j.

Similarly,

$$HB(y_i^*) = \{x^{**} \in B_{E^{**}}(0,1) : x^{**}(y_j^*) = 1\}$$

is a weak*-closed face with $z_i \in HB(y_i^*)$.

Thus we may replace each z_i by elements in ext $HB(y_i^*)$. This completes the proof. ∎

In the next section we shall need to know that some functionals have a unique norm-preserving extension from $R(E,F)$ or $K(E,F)$ to $L(E,F)$. We start with some simple results.

LEMMA 9: *If $\phi = x^{**} \otimes x^* \in B_{E^{**}}(0,1) \otimes B_{E^*}(0,1)$ with $x^{**}(x^*) = 1$ and $\hat{\phi}$ is the canonical extension from $R(E)$ to $X = R(E) \oplus R \cdot I$, then*

$$\hat{\phi} \in \text{ext} HB(\phi).$$

Proof: If $\hat{\phi} \notin$ ext $HB(\phi)$, then we can find $\eta \in X^*$, $\eta \neq 0$, such that

$$\hat{\phi} \pm \eta \in HB(\phi).$$

But then $\eta \in R(E)^{\perp}$. Moreover, we have

$$1 \geq (\hat{\phi} \pm \eta)(I) = 1 \pm \eta(I)$$

so $\eta(I) = 0$. But then $\eta = 0$ and we have $\hat{\phi} \in$ ext $HB(\phi)$. ∎

LEMMA 10: *Every $\phi \in$ ext $B_{R(E,F)^*}(0,1)$* has a unique norm-preserving extension to $K(E,F)$.

Proof: Let $\phi \in$ ext $B_{R(E,F)^*}(0,1)$ and assume $\phi_1, \phi_2 \in$ ext $HB(\phi)$. Since $HB(\phi)$ is a weak*-closed face, there exist, by a theorem of Ruess and Stegall [20], $x_i^{**} \in$ ext $B_{E^{**}}(0,1)$ and $y_i^* \in$ ext $B_{F^*}(0,1)$ such that $\phi_i = x_i^{**} \otimes y_i^*$ for $i = 1, 2$.

Let $T \in K(E,F)$. We shall show that $\phi_1(T) = \phi_2(T)$.

Assume first that y_1^* and y_2^* are linearly dependent. Then we may assume that $y_1^* = y_2^*$. Choose $y \in F$ such that $y_i^*(y) = 1$ and define $S \in R(E,F)$ by

$$S = T^* y_i^* \otimes y.$$

Then

$$\phi_1(T) = \phi_1(S) = \phi(S) = \phi_2(S) = \phi_2(T).$$

Assume next that y_1^* and y_2^* are linearly independent. We can find $y_1, y_2 \in F$ such that

$$0 = y_2^*(y_1) = y_1^*(y_2),$$
$$1 = y_2^*(y_2) = y_1^*(y_1).$$

Define $S \in R(E,F)$ by

$$S = T^* y_1^* \otimes y_1 + T^* y_2^* \otimes y_2.$$

Then

$$\phi_1(T) = \phi_1(S) = \phi(S) = \phi_2(S) = \phi_2(T).$$

Thus ext $HB(\phi)$ consists of a unique point, and ϕ has a unique norm-preserving extension to X. ∎

In the next section, it is crucial to know that sufficiently many functionals have unique norm-preserving extensions from $R(E)$ or $K(E)$ to $L(E)$, or at least to a subspace of $L(E)$ containing I. The first of the results in this direction is the following lemma. Ruess and Stegall first proved this result in Theorem 4 in [22]. (See also [21] for similar results.) Our proof is similar to the proof of Lemma 5.1 of Godefroy and Saphar [7] and uses a result of D. Werner [23].

LEMMA 11: *Let $\phi = x \otimes y^* \in$ dent $B_E(0,1) \otimes w^*$−dent $B_{F^*}(0,1)$. Then $\hat{\phi} \in$ dent $B_{L(E,F)^*}(0,1)$ and ϕ has a unique norm-preserving extension from $R(E,F)$ to $L(E,F)$.*

Proof: $HB(\phi)$ is a weak*-closed face of $B_{L(E,F)^*}(0,1)$ containing the natural extension $\hat{\phi}$ of ϕ. Let $\psi \in HB(\phi)$. We shall show that $\psi = \hat{\phi}$.

Let $\epsilon > 0$ and let $T \in L(E,F)$ with $\|T\| = 1$ such that

$$\|\psi - \hat{\phi}\| < \epsilon + |\hat{\phi}(T) - \psi(T)|.$$

By Lemma 4.5 of D. Werner [23], to every $\partial \in \langle 0, \epsilon \rangle$ sufficiently small, we can find $x^* \in E^*$ such that

$$\begin{aligned}
&x^*(x) = 1,\\
&\|x^*\| \leq 1 + \epsilon \cdot \partial,\\
&x^*(z) > 1 - \partial \quad \text{and} \quad \|z\| \leq 1 \Rightarrow \|z - x\| \leq \epsilon.
\end{aligned}$$

Similarly, we can find $y \in F$ such that

$$\begin{aligned}
&y^*(y) = 1,\\
&\|y\| \leq 1 + \epsilon \cdot \partial,\\
&z^*(y) > 1 - \partial \quad \text{and} \quad \|z^*\| \leq 1 \Rightarrow \|z^* - y^*\| \leq \epsilon.
\end{aligned}$$

Let $S = x^* \otimes y \in R(E,F)$. Then $\phi(S) = 1$.

As in the proof of Theorem 8, we can find m, $\lambda_i > 0$, $x_i \in B_E(0,1)$ and

$y_i^* \in B_{F^*}(0,1)$ such that

$$\sum_{i=1}^{m} \lambda_i = 1,$$

$$|\phi(S) - \sum_{i=1}^{m} \lambda_i y_i^*(Sx_i) < \partial^2,$$

$$|\psi(T) - \sum_{i=1}^{m} \lambda_i y_i^*(Tx_i)| < \epsilon.$$

Let $\alpha = \partial(1 - \epsilon + \epsilon\partial)$ and let

$$J_0 = \{i : y_i^*(Sx_i) \leq 1 - \alpha\}.$$

Since $1 = \phi(S) = \psi(S)$, we get

$$\begin{aligned} 1 - \partial^2 &< \sum_{i=1}^{m} \lambda_i y_i^*(Sx_i) \\ &\leq \sum_{i \in J_0} \lambda_i(1-\alpha) + \sum_{i \notin J_0} (1+\epsilon\partial)^2 \\ &= (1 + 2\epsilon\partial + \epsilon^2\partial^2) - (\alpha + 2\epsilon\partial + \epsilon^2\partial^2) \sum_{i \in J_0} \lambda_i. \end{aligned}$$

From this it follows that $\sum_{i \in J_0} \lambda_i < 3\epsilon$.

From $i \notin J_0$, we have

$$y_i^*(Sx_i) = y_i^*(y) \cdot x^*(x_i) > 1 - \alpha.$$

So

$$1 - \alpha < y_i^*(y) \cdot (1 + \epsilon\partial)$$

and

$$y_i^*(y) > (1-\alpha)/(1+\epsilon\partial) = 1 - \partial.$$

Similarly for $i \notin J_0$,

$$x^*(x_i) > 1 - \partial.$$

Hence, $\|y_i^* - y^*\| \leq \epsilon$ and $\|x_i - x\| \leq \epsilon$ for $i \notin J_0$.

But then we have

$$\begin{aligned}\|\psi-\hat{\phi}\| &< \epsilon + |\psi(T)-\hat{\phi}(T)| \\ &< \epsilon + \Big|\psi(T)-\sum_{i=1}^{m}\lambda_i y_i^*(Tx_i)\Big| + \Big|\hat{\phi}(T)-\sum_{i=1}^{m}\lambda_i y_i^*(Tx_i)\Big| \\ &< 2\epsilon + \sum_{i=1}^{m}\lambda_i|y^*(Tx)-y_i^*(Tx_i)| \\ &< 2\epsilon + \sum_{i\in J_0}\lambda_i 2(1+\epsilon\partial) + \sum_{i\notin J_0}\lambda_i(|y_i^*(Tx-Tx_i)| + |(y^*-y_i^*)(Tx_i)|) \\ &< 2\epsilon + 6\epsilon(1+\epsilon^2) + 2\epsilon < 16\epsilon.\end{aligned}$$

Thus $\psi = \hat{\phi}$.

The same argument shows that if we start with ψ such that $\|\psi\| \leq 1$ and $\psi(S) > 1-\partial^2$, then we get $\|\psi-\hat{\phi}\| < 16\epsilon$. Thus $\hat{\phi}$ is a denting point. ∎

The above result is what we need to prove Theorem 0 for reflexive spaces. To extend Theorem 0 to non-reflexive spaces, we need to have a similar theorem when not both E and E^* contain denting or strongly exposed point. We have only a partial result in this direction.

LEMMA 12: *Assume $x \in B_E(0,1)$ is strongly exposed by $x^* \in B_{E^*}(0,1)$.*

Let $\phi = x \otimes x^* \in R(E)^*$. Then ϕ has a unique norm-preserving extension to $X = R(E) \oplus R \cdot I$.

Proof: Let $\hat{\phi} = x \otimes x^*$ be the natural extension of ϕ to X, and let $\psi \in HB(\phi)$. We shall show that $\psi = \hat{\phi}$.

Let $S = x^* \otimes x \in R(E)$. Then we have

$$1 = \phi(S) = \psi(S)$$

and

$$1 = \hat{\phi}(I).$$

We shall prove that $\psi(I) = 1$.

Let $m, \lambda_i > 0$, $x_i \in B_E(0,1)$ and $x_i^* \in B_{E^*}(0,1)$ be such that

$$\sum_{i=1}^{m}\lambda_i = 1,$$

$$\Big|\psi(S) - \sum_{i=1}^{m}\lambda_i x_i^*(Sx_i)\Big| < \partial^2$$

and

$$\left|\psi(I) - \sum_{i=1}^{m} \lambda_i x_i^*(Ix_i)\right| < \epsilon.$$

We have chosen $\partial > 0$ so small that if $\|z\| \leq 1$ and $x^*(z) \geq 1-\partial$ then $\|x-z\| \leq \epsilon$.

Put

$$J_0 = \{i : x_i^*(Sx_i) \leq 1 - \partial\}.$$

Then we have

$$\begin{aligned} 1 - \partial^2 &< \sum_{i=1}^{m} \lambda_i x_i^*(Sx_i) \\ &\leq \sum_{i \in J_0} \lambda_i (1 - \partial) + \sum_{i \notin J_0} \lambda_i \\ &= 1 - \partial \sum_{i \in J_0} \lambda_i. \end{aligned}$$

Thus

$$\sum_{i \in J_0} \lambda_i \leq \partial.$$

For $i \notin J_0$, we have

$$\begin{aligned} 1 - \partial \leq x_i^*(Sx_i) &= x_i^*(x) x^*(x_i) \\ &\leq x^*(x_i). \end{aligned}$$

(We can assume $x^*(x_i) \geq 0$ for all i.)

Thus for $i \notin J_0$, we have

$$x_i^*(x) \geq 1 - \partial$$

and

$$\|x_i - x\| \leq \epsilon.$$

Hence we get

$$\begin{aligned} |\hat{\phi}(I) - \psi(I)| &\leq \left|\psi(I) - \sum_{i=1}^{m} \lambda_i x_i^*(Ix_i)\right| + \left|\hat{\phi}(I) - \sum_{i=1}^{m} \lambda_i x_i^*(Ix_i)\right| \\ &\leq \epsilon + \left|1 - \sum_{i=1}^{m} \lambda_i x_i^*(x_i)\right| \\ &\leq \epsilon + \sum_{i \in J_0} \lambda_i |1 - x_i^*(x_i)| + \sum_{i \notin J_0} \lambda_i |(1 - x_i^*(x)) + x_i^*(x - x_i)| \\ &\leq \epsilon + 2\partial + \sum_{i \notin J_0} \lambda_i (\partial + \epsilon) \\ &\leq 2\epsilon + 3\partial. \end{aligned}$$

This shows that $\psi(I) = \hat{\phi}(I) = 1$. ∎

4. The approximation property

In this section, we shall combine the results from the previous sections with results by J. Johnson [11] to obtain characterizations of spaces with the metric approximation property. We shall write AP for the approximation property. Results on the approximation property can be found in [18].

J. Johnson [11] shows how we can use the AP to obtain the existence of projections. We shall use the existence of certain projections to show that the space has the metric AP. The results come in two versions, one with the metric AP and one with the metric compact AP.

THEOREM 13: *Assume that F is a Banach space with the Radon–Nikodym property. Then the following statements are equivalent:*

(1) *F has the metric AP.*
(2) *$R(F)^{\perp}$ is the kernel of a norm-one projection in $L(F)^*$.*
(3) *$R(E,F)^{\perp}$ is the kernel of a norm-one projection in $L(E,F)^*$ for any Banach space E.*
(4) *$R(F)^{\perp\perp}$ is the image of a norm-one projection in $L(F)^{**}$.*
(5) *$R(E,F)^{\perp\perp}$ is the image of a norm-one projection in $L(E,F)^{**}$ for any Banach space E.*
(6) *There exists an isometry $T : L(F) \to R(F)^{**}$ such that the restriction to $R(F)$ is the canonical imbedding.*
(7) *There exists an isometry $T : L(E,F) \to R(E,F)^{**}$ such that the restriction to $R(E,F)$ is the canonical imbedding for any Banach space E.*
(8) *$\bar{R}(F)$ has the $n.L(F)$.I.P. for all n.*
(9) *$\bar{R}(E,F)$ has the $n.L(E,F)$.I.P. for all n and any Banach space E.*
(10) *$\bar{R}(F)$ has the $n.X$.I.P. for all n when $X = \bar{R}(F) \oplus R \cdot I$.*

Proof: (1) $\Rightarrow$ (7) is proved in J. Johnson [11].
(7) $\Rightarrow$(6) $\Rightarrow$(2) $\Rightarrow$(4) and (7) $\Rightarrow$(3) $\Rightarrow$(5) $\Rightarrow$(4) follow from Theorem 1 and "the principle of local reflexivity".
(5) $\Rightarrow$(9) $\Rightarrow$(8) $\Rightarrow$(10) and (4) $\Rightarrow$(8) are easy.
We shall now show that (10) $\Rightarrow$(1).

Note that $\bar{R}(F)$ is of co-dimension one in $X = \bar{R}(F) \oplus R \cdot I$.

By Theorem 1 and Proposition 3.2 in [15], we have $R(F)^{\perp} = \ker P$ for a norm-one projection P in X^*. We have

$$P^*I \in R(F)^{\perp\perp} \subset X^{**}$$

and $\|P^*I\| \leq 1$. Thus there exists a net $(T_\alpha) \subset R(F)$ such that $\|T_\alpha\| \leq 1$ for all α and

$$T_\alpha \to P^*I$$

weak* in X^{**}.

Let $x \in B_F(0,1)$ be strongly exposed by $x^* \in B_{F^*}(0,1)$, and let $\phi = x \otimes x^*$.

By Lemma 12, ϕ has a unique norm-preserving extension to X. Thus if $\hat{\phi}$ is the canonical extension of ϕ to X, then $P\hat{\phi} = \hat{\phi}$.

Thus

$$\phi(T_\alpha) \to (P^*I)(\hat{\phi}) = (P\hat{\phi})(I) = \hat{\phi}(I)$$

such that

$$x^*(T_\alpha x) \to x^*(x) = 1.$$

But then $\|T_\alpha x - x\| \to 0$.

Since

$$B_F(0,1) = \overline{\text{conv}}(\text{str.exp} B_F(0,1))$$

(norm-closure), we get that $\|T_\alpha x - x\| \to 0$ for all $x \in F$. ∎

THEOREM 14: *Theorem* 13 *is true if we make the following modifications:*

(i) In (1), F has the metric compact AP.

(ii) $\bar{R}(F)$ and $\bar{R}(E,F)$ are everywhere replaced by $K(F)$ and $K(E,F)$.

THEOREM 15: *Assume E and F are reflexive spaces. The following statements are equivalent:*

(1) *$R(E,F)^{\perp}$ is the kernel of a norm-one projection in $K(E,F)^*$.*

(2) *$\bar{R}(E,F)$ has the $n.K(E,F)$.I.P. for all n.*

(3) *$\bar{R}(E,F) = K(E,F)$.*

Proof: (3) ⇒(1) ⇒(2) are trivial.

(2) ⇒(3). Let $T \in K(E,F)$. Since $T(E)$ is a separable subspace of F, there exists [17] a separable subspace X with $T(E) \subseteq X \subseteq F$ and a norm-one projection on F with image X. But then $\bar{R}(E,X)$ has the $n.K(E,X)$.I.P. for all n. Hence we can and shall assume that F is separable. Similarly, we can assume E is separable.

Let $Y = \bar{R}(E,F) + R \cdot T$. By (2) there exists a projection P in Y^* such that $R(E,F)^{\perp} = \ker P$ and $\|P\| = 1$. By Lemma 10, for each $\phi \in \text{ext } B_{Y^*}$, we have $P\phi = \phi$. Now $P^*T \in R(E,F)^{\perp\perp}$ and since E and F are separable, there exists a sequence (T_n) in $R(E,F)$ such that $\|T_n\| = \|P^*T\|$ for all n and

$$T_n \to P^*T$$

weak* in Y^{**}. For every $\phi \in \text{ext } B_{Y^*} = \text{ext } B_{K(E,F)^*}$, we get

$$\lim_n \phi(T_n) = P^*T(\phi) = (P\phi)(T) = \phi(T).$$

Thus $T_n \to T$ pointwise on ext $B_{K(E,F)^*}$. By Rainwater's theorem [4] this implies that $T_n \to T$ weakly in $K(E,F)$. But then $T \in \bar{R}(E,F)$. ∎

Remarks:

(a) In the proofs of Theorems 13 and 14, we used that F has the Radon–Nikodym property in the proof of (10) $\Rightarrow$(1). The important thing is that the unit ball of F equals the *norm-closure* of the convex hull of its strongly exposed points.

(b) If $K(F)$ is an M-ideal in $X = K(F) \oplus R \cdot I$, then it follows that F has the metric compact AP. From Theorem 2.2 in Lima [15], it follows that if $K(F)$ is an M-ideal in X, then F is an M-ideal in F^{**}, so F^* has the Radon–Nikodym property. See also Kalton [13].

It is well known that if a Banach space F^* has the metric AP, then also F has the metric AP. For the intersection property, we have a similar result.

PROPOSITION 16: *If $\bar{R}(F^*)$ has the $n.L(F^*)$.I.P., then $\bar{R}(F)$ has the $n.L(F)$.I.P.*

Proof: Let $S_1, \ldots, S_n \in R(F)$, let $S_0 \in L(F)$ and let $r_i = \|S_0 - S_i\|$. Let $\epsilon > 0$ and

$$S \in R(F^*) \cap \bigcap_{i=1}^{n} B(S_i^*, r_i + \epsilon).$$

Let T be the restriction of S^* to F.

Then $T : F \to F^{**}$ and

$$\|S_i - T\| \leq r_i + \epsilon, \quad i = 1, \ldots, n.$$

Since T and every S_i has finite rank, we can use "the principle of local reflexivity" to show that

$$R(F) \cap \bigcap_{i=1}^{n} B(S_i, r_i + 2\epsilon) \neq \emptyset.$$

∎

PROBLEM 2: *Is Proposition* 16 true for the space of compact operators?

For dual spaces we have the next result.

THEOREM 17: *Assume F^* has the Radon–Nikodym property. Then the following statements are equivalent:*

(1) *There exists a net (T_α) in $R(F)$ such that*
$\|T_\alpha\| \leq 1$ *for all* α,
$\|T_\alpha x - x\| \to 0$ *for all* $x \in F$,
$\|T_\alpha^* x^* - x^*\| \to 0$ *for all* $x^* \in F^*$.
(2) F^* *has the metric* AP.
(3) *There exists an isometry $T : L(F^*) \to R(F^*)^{**}$ such that the restriction to $R(F^*)$ is the natural imbedding.*
(4) $R(F^*)^\perp$ *is the kernel of a norm-one projection in* $L(F^*)^*$.
(5) $R(F^*)^{\perp\perp}$ *is the image of a norm-one projection in* $L(F^*)^{**}$.
(6) $\bar{R}(F^*)$ *has the* $n.L(F^*)$.I.P. *for all* n.
(7) $\bar{R}(F^*)$ *has the* $n.X$. *I.P. for all n where $X = \bar{R}(F^*) \oplus R \cdot I$.*

Proof: Most of the proof is similar to the proof of Theorem 13.

Note that (1) $\Rightarrow$(2) is trivial.

(7) $\Rightarrow$(1). As in (10) $\Rightarrow$(1) in Theorem 13, assuming (7) here, we can find a net (T_α) in $R(F^*)$ such that $\|T_\alpha\| \leq 1$ for all α and

$$\|T_\alpha x^* - x^*\| \to 0$$

for all $x^* \in F^*$.

We shall show that we may assume $T_\alpha = S_\alpha^*$ where $S_\alpha \in R(F)$. From this (1) follows arguing as in the proof of Lemma 5.1 in [9].

Let $x_1^*, \ldots, x_m^* \in B_{F^*}(0,1)$ be strongly exposed by $x_1^{**}, \ldots, x_m^{**}$ in $B_{F^{**}}(0,1)$.

Let $\epsilon > 0$ and let $\partial > 0$ such that if $\|x^*\| \leq 1$ and $x_i^{**}(x^*) > 1 - \partial$ for some i, then $\|x^* - x_i^*\| < \epsilon$.

Choose T_α such that

$$|1 - x_i^{**}(T_\alpha x_i^*)| < \partial \quad \text{for all } i.$$

Since T_α has finite rank, we can use "the principle of local reflexivity" to find $S_\alpha \in R(F)$ such that

$$|1 - x_i^{**}(S_\alpha^* x_i^*)| < \partial \quad \text{for all } i.$$

Thus $\|S_\alpha^* x_i^* - x_i^*\| < \epsilon$ for all i. This completes the proof. ∎

THEOREM 18: *Assume F^* has the Radon–Nikodym property. The following statements are equivalent:*

(1) *F^* has the metric* AP.
(2) *There exists an isometry $T : L(F,E) \to R(F,E)^{**}$ such that the restriction to $R(F,E)$ is the natural map for every Banach space E.*
(3) *$R(F,E)^\perp$ is the kernel of a norm-one projection in $L(F,E)^*$ for every Banach space E.*
(4) *$R(F,E)^{\perp\perp}$ is the image of a norm-one projection in $L(F,E)^{**}$ for every Banach space E.*
(5) *$\bar{R}(F,E)$ has the $n.L(F,E)$.I.P. for every Banach space E.*
(6) *There exists an isometry $T : L(F,F^{**}) \to R(F,F^{**})^{**}$ such that the restriction to $R(F,F^{**})$ is the natural map.*
(7) *$R(F,F^{**})^\perp$ is the kernel of a norm-one projection in $L(F,F^{**})^*$.*
(8) *$R(F,F^{**})^{\perp\perp}$ is the image of a norm-one projection in $L(F,F^{**})^{**}$.*
(9) *$\bar{R}(F,F^{**})$ has the $n.L(F,F^{**})$.I.P. for all n.*
(10) *$\bar{R}(F,F^{**})$ has the $n.X$.I.P. for all n when $X = \bar{R}(F,F^{**}) \oplus R \cdot I_F$.*

Proof: (1) ⇒(2). Let (T_α) be a net in $R(F)$ as in (1) in Theorem 17. Taking a subnet, if necessary, we may assume that $\lim \phi(T_\alpha)$ exists for all $\phi \in R(F)^*$. As Johnson [11] has proved, this implies that the operator T defined by

$$T(S)(\phi) = \lim \phi(ST_\alpha),$$

where $\phi \in R(F,E)^*$ and $S \in L(F,E)$, satisfies the requirements in (2).
(2) ⇒(3) ⇒(4) ⇒(5) ⇒(9) and
(2) ⇒(6) ⇒(7) ⇒(8) ⇒(9) ⇒(10) are easy.
(10)⇒(1). By Theorem 17, it suffices to show that $\bar{R}(F^*)$ has the $n.X$.I.P. for all n when $X = \bar{R}(F^*) \oplus R \cdot I$. In order to do that, let $\epsilon > 0$, let $S_1, \dots, S_n \in R(F^*)$ and let $r_i = \|S_i - I\|$ for $i = 1, \dots, n$. It suffices to show that there exists $S \in R(F^*)$ such that $\|S - S_i\| < r_i + \epsilon$ for all i.

We have $\|S_i^*|_F - I_F\| \le r_i$ for all i. By (10), we can find $U \in R(F,F^{**})$ such that $\|S_i^*|_F - U\| \le r_i + \epsilon$ for all i. Now, let S be the restriction to F^* of U^*. ∎

Theorem 14 says that Theorem 13 is true also for compact operators. Theorems 17 and 18 have a partial generalization to compact operators.

THEOREM 19: *Assume F^* has the Radon–Nikodym property. Then statements* (2)–(7) *in Theorem* 17 *and statements* (1)–(10) *in Theorem* 18 *are all equivalent if we replace metric* AP *by metric compact* AP *and replace finite rank operators by compact operators everywhere.*

Proof: That (2)–(7) in Theorem 17 are equivalent follows from Theorem 14. In order to prove that statements (1)–(10) in Theorem 18 are equivalent for compact operators, we only have to show that if $K(F, F^{**})$ has the $n.X$.I.P. when $X = K(F, F^{**}) + R \cdot I_F$, then F^* has the metric compact AP. This can be done in the same way as (10) $\Rightarrow$(1) in Theorem 13 was proved. But we need a version of Lemma 12 that takes care of the unique extension of enough functionals.

Let $x^* \in B_{F^*}(0,1)$ be strongly exposed by $x^{**} \in B_{F^{**}}(0,1)$ and $\phi = x^{**} \otimes x^* \in K(F, F^{**})^*$. We shall show that ϕ has a unique extension to X. This will suffice since $\phi(I) = 1$.

Let $S = x^* \otimes x^{**} \in K(F, F^{**})$. Now, using that conv $(B_F(0,1) \otimes B_{F^*}(0,1))$ is weak* dense in $B_{L(F,F^{**})^*}(0,1)$, we can proceed as in Lemma 12 to show that ϕ has a unique extension to X. ∎

The recent paper [8] by Godefroy, Kalton and Saphar contains many interesting results related to the results in this paper. The reader should consult that paper, especially chapter 8.

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